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STUDENTS' INFORMAL MEASURES BETWEEN OBJECTS AND TOOLS

Christian Büscher

TU Dortmund University

Drawing informal statistical inferences is often seen as the main vehicle for meaningful statistics education, and informal statistical inference is grounded on the adequate use of statistical measures to connect data with claims. This raises the question of how learners can participate in this informal inferential reasoning before having access to statistical measures by formal statistics education. This paper introduces the notion of informal statistical measures in order to draw on learners' statistical conceptions prior to learning about regular statistical measures in class. A case study stemming from a larger design research project analyzes the learning processes of two students on a micro level. The interpretative analysis shows that the roles of the students' informal measures repeatedly switch between object and tool of investigation, and these switches allow the students to expand the considered characteristics and uses of informal measures. Possible connections to develop these informal measures into regular statistical measures are drawn up.

INTRODUCTION

One of the main goals of statistics education is enabling students to make data-based claims about an unknown, wider universe beyond the data at hand, called (Informal) Statistical Inference (Makar & Rubin, 2009). In order to make informal statistical inferences, students have to draw on a wide area of knowledge, including knowledge of statistical concepts and habits of looking at and organizing data to support their reasoning (Makar, Bakker, & Ben-Zvi, 2011). On the other hand, students develop meaning for these statistical concepts by utilizing them in making statistical inferences (Bakker & Derry, 2011). At the beginning of their statistical education, learners have only little knowledge of statistically valid inferences. Hence they have to resort to individual ways of tackling statistical tasks such as comparing groups. These individual ways of reasoning often show promising individual statistical conceptions that touch on a wide range of regular statistical concepts (Schnell & Büscher, 2015). It is the task of statistics education to connect these individual conceptions to regular statistical concepts in order to let individual ways of reasoning mature into more formal ways of drawing statistical inferences.

INFORMAL AND REGULAR MEASURES

Drawing informal statistical inferences is commonly seen as a way to enable students to participate in the 'real use' of statistics without having to understand the complex mathematical background of formal inferential statistics. These informal inferences still have to be based on data, which often means that statistical measures such as median and range have to be employed. Learners without specific instruction however have to use natural language instead of measures to talk about data. This natural language is rich in concepts, as many different statistical concepts can be encapsulated in a single expression (Makar & Confrey, 2005). This stands in contrast to statistical measures, which provide highly sophisticated means to talk about specific statistical concepts. The question remains of how learners' use of concept-rich natural language can be used as a starting point for developing more formal ways of reasoning.

In order to design learning pathways from learners' individual conceptions to regular statistical concepts, this study conceptualizes learners' natural language about data as *informal* statistical measures corresponding to *individual* statistical conceptions (Fig. 1). Examples for such informal measures can be modal clumps (Konold et al., 2002) or partition into chunks of high-middle-low (Makar & Confrey, 2003). Contrary to *regular* statistical measures, informal measures correspond to individual conceptions such as representativeness, typicality or normality (Makar, 2014; Büscher, 2016), that simultaneously capture aspects of many different statistical concepts. Thus, a modal clump used for talking about location and density of 'the majority of the data' is as much a measure as the standard deviation is for talking about variability. The focus of this study is to track the use of learners' informal measures, and to show possible connections from informal to regular measures in order to support the development of individual conceptions to statistical concepts.

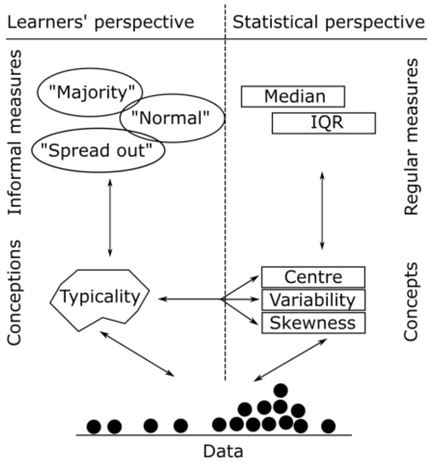

Fig. 1: The Interplay of concepts and conceptions

MATHEMATICAL CONCEPTS AS OBJECTS AND TOOLS

To a statistician, regular measures do not only function as tools for specific statistical questions, but also have characteristics of objects. Regular measures can be studied on their own, revealing their properties regarding behavior with certain kinds of distributions, and can be embedded within larger statistical procedures. This dual uses of concepts as tools and objects forms the basis of the tool-object dialectic of mathematical concepts (Douady, 1985; Artigue, 1995). Mathematical concepts gain their meaning first in the development as tools for solving problems not possible to solve adequately with old knowledge. Later they are institutionalized as objects on their own, which then can be part of new problems calling for development of new tools.

This study adopts the approach of the tool-object dialectic on a micro level in order to track the learning processes of students consolidating their use of informal measures. This raises questions on the exact nature of this dialectic, for example whether there is a clear switch from tool to object or there is a cyclic progression of switches. With the learner in the active role of developing tools for tackling problems, it also calls for careful consideration of problems suitable for developing individual conceptions into statistical concepts.

The research interest of the presented case study is to gain deeper insights into how grade 7 students without previous formal statistics education use informal measures to make informal inferences. Since these informal measures touch on many aspects of statistical concepts like central tendency and variability, they can be considered as promising indications of resources for building more formal ways of reasoning. The presented study from a larger design research project investigates students' use of informal measures as this can provide access to their individual conceptions, and show possible ways of supporting their learning processes through adequate task design. Thus, the general research interest is focused on the following research questions:

(RQ1) What informal measures do students base their inferences on, and what kind of inferences do they draw?

(RQ2) How do these informal measures develop in their dual roles of object and tool?

RESEARCH DESIGN

This study adopts the methodological framework of topic-specific didactical design research (Prediger et al., 2012) with a focus on learning processes (Prediger, Gravemeijer, & Confrey, 2015). Through iterative cycles of design experiments, this approach aims at providing empirically grounded local theories on topic-specific learning processes as well as design principles and concrete teaching-learning arrangements for the topic. This study reports on the third cycle of design experiments.

Data gathering

In order to gain insight into students' informal measures prior to formal statistics education, design experiments were conducted with five pairs of grade 7 students (aged 12 – 14) who had not yet undergone statistics teaching within the mathematics classroom. Each experiment consisted of two sessions of 45 minutes each, with each session having its own arrangements of data and tasks. Experiments were fully videotaped and partially transcribed. The process-focused analysis of the video data from the second session of the third cycle of design experiments allows to reconstruct the development of students' informal measures and their interplay with the task design.

Task design

For designing the teaching-learning arrangements, various design principles have been implemented throughout the different cycles of design research, two of which play an important part in this study.

Drawing on individual conceptions of typicality. Among the conceptions underlying the concept-rich language of learners are those of typicality and representativeness. Whereas Makar (2014) identified them as being in relation to the mean, previous cycles of the design research study revealed evidences for individual conceptions of representativeness (Schnell & Büscher, 2015) and of typicality (Büscher, 2016) not referring to the mean. Rather, 'typical' or 'normal' were used by students to partition distributions into sections of exception and rule, characterized by an area in the middle of a distribution reminiscent of the box of a box plot. 'Typical' provided the students with a language for talking about phenomena relating to concepts of central tendency and variability. This design principle builds on results from previous cycles and provides the starting point for initiating learning processes that draw develop individual conceptions towards regular concepts.

Communicating about critical life-world phenomena. To allow learners to build on their individual conceptions, the problem to be presented has to be carefully considered. Three features for such problems have been identified in previous cycles to support the conceptual development: For drawing on the concept-rich natural language, the problem and the teaching-learning arrangement should be structured in a *communicative* way. Communication about *life-world phenomena* (rather than about statistical concepts) facilitates meaningful discussions. And the problem has to be *critical*, in that failure to communicate would have relatable life-world consequences.

In order to create the need for reliable inferences based on data, the task design put the students into the role of researchers of climate change. Students were given distributions of monthly lowest arctic sea ice extent for the years 1982, 1992, and 2012 (Fig. 2; data was taken from Fetterer, Knowles, Meier, & Savoie, 2002 and slightly modified for didactical purposes) and were asked to give a report whether, and how much, the ice area had changed.

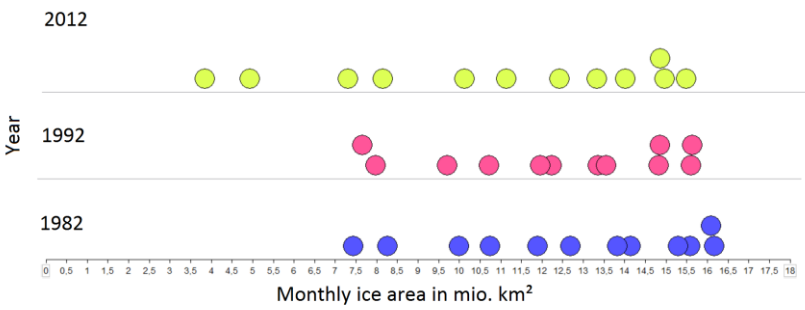

Fig. 2: Distribution of sea ice extent in session 2 (Translated from German).

To facilitate discussion, and to offer a range of different measures the students were given filled-in 'report sheets' (Fig. 3) of fictional students claiming either overall change (because the spread had increased) or no significant change (because the modal clump was still high in 2012). These differing interpretations were based on different individual definitions of the measures employed. After discussing the report sheets, the students were asked to correct them and to create their own. This raised the question which measures should be used and what these measures actually stand for.

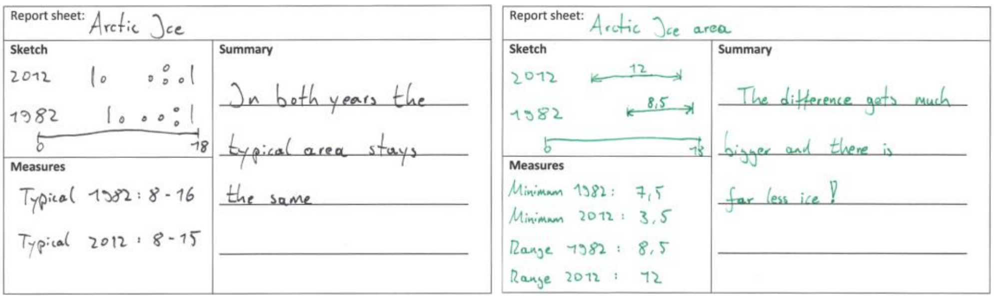

Fig. 3: Fictitious students' filled-in report sheets (Translated from German)

It is important to note that although the measures 'Typical', 'Minimum' and 'Range' employed in the filled-in report sheets were not invented by all students prior to the work on the filled-in sheets, they were introduced without any explanation at all. The students could not consult any definition and no explanation was given by the researcher what the fictitious students could have meant. This was even the case with the measure 'Minimum', which in previous cycles was not always interpreted as absolute minimum, but sometimes as lowest 'reasonable' value. Thus, these measures can still be counted as informal measures of individual conceptions, because (1) these measures do not necessarily correspond to the regular minimum and range, (2) students still needed to construct their own meaning for these measures, and (3) these measures correspond to individual conceptions found in previous cycles. In this way, the design principles were implemented by initiating communication about the melting arctic sea ice using informal measures.

Data was presented to the students in TinkerPlots (Konold & Miller, 2011). Because informal measures are not necessarily possible to construct using TinkerPlots' tools for manipulating data, a screen overlay software was used that allowed students to freely draw their informal measures on the screen. The experiments however did not make use of TinkerPlots' own functionalities.

Data Analysis

In order to track the students' individual conceptions and the role of informal measures as tools and objects, the transcribed data was analyzed under an interpretative paradigm using concepts-in-action and theorems-in-action from Vergnaud's theory of conceptual fields (Vergnaud, 1996). Concepts-in-action are "categories (objects, properties, relationships, transformations, processes etc.) that enable the subject to cut the real world into distinct elements and aspects, and pick up the most adequate selection of information according to the situation and scheme involved" (Vergnaud, 1996, p. 225). Theorems-in-action are statements held to be true by the learner. They are intricately connected to the learners' concepts-in-action, as theorems-in-action give meaning to concepts-in-action, which in turn give content to the theorems-in-action. Concepts-in-action and theorems-in-action are reconstructed from the students' point of view, and do not necessarily correspond to regular statistical concepts. Thus, reconstructing the students' concepts-in-action provides access to their individual conceptions organizing their actions. In the analysis, the reconstructed concepts-in-action are symbolized by ||…||, while theorems-in-action are denoted by <…>.

SNAPSHOT: STUDENTS' INVENTION OF TYPICAL

The following empirical snapshot follows the 13 year old students Quanna (Q) and Rebecca (R), and the researcher (I), in their struggle to find the meaning and computation for the informal measure 'Typical' (in the following sections indicated through use of upper-case T). The excerpts stem from a conversation of about 15 minutes. The snapshot starts with the students filling out their own report sheet after they have discussed the filled-in report sheets.

1 Q *[while filling out own report sheet]* And Typical…

2 R Typical […] it could be, like, the middle or something?

3 R I would say the middle and a bit higher.

Measures such as Minimum and Range do not seem sufficient for capturing the relevant aspects of the phenomenon of arctic sea ice decline to the students; ||*Typical*|| is the main category organizing their view on the situation. Since the measure itself is still ill-defined, they refer to another feature of the distribution to place it, the ||middle|| of the distribution (it remains unclear what exactly is meant by 'middle'). ||*Typical*|| however is not a tool for determining the ||*middle*||, but being near the ||*middle*|| is a characteristic of ||*Typical*||: *<Typical is located a bit higher than the middle>*. Typical is not seen as a tool, but rather as the object under investigation.

Some minutes later, the students are about to write their summary for the report sheet.

20 Q Okay, now the summary.

21 R The numbers got *[points to own report sheet]* – look – more ice melted away.

22 Q *[shakes head]* the difference is – is around 2.5.

23 R Always?

24 Q Yes, right here *[points to own report sheet]* of Typical.

Rebecca seems to have difficulties with combining the life-world phenomenon (the melting ice) with the task of giving a short data-backed summary. At this point, Quanna is able to utilize their measure of Typical. In the meantime, the students have decided that *<Typical is a number>*, which they intuitively identified for the distributions of 1982 and 2012 by 11 and 13.5 and noted on their report sheet (Fig. 4; later changed to 13). These numbers show a difference of 2.5, which can now be used in their summary to talk about arctic sea ice decline: <*Typical can be used to describe differences in distributions of arctic sea ice*>. This does not necessarily imply that Typical is seen as a tool in a statistical investigation to decide whether arctic sea ice declined, as this decline was seen as given from the beginning of the task. Still, Typical functions as a tool for talking about differences found during ||*comparison*|| of distributions.

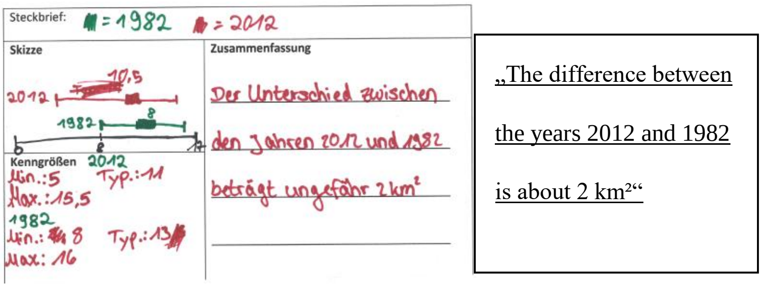

Fig. 4: Quanna and Rebecca's report sheet

The definition of Typical still being unclear, the researcher challenges them to explain their use.

40 I I see you decided to use only one number for Typical, in contrast to this report sheet, where they used an area *[points to filled-in report sheet]*. Is that better or worse, what do you think?

41 R Well Typical is more of a single…

42 Q *[simultaneously]* more of an area…

43 R Now we disagree. […] Typical is more of a small area, or you could say a number. Like here, from 10 to 12. […] If the area is over 100, it may be over 10. […] But never more than the half.

The claim <*Typical is a number*> becomes disputed, as ||*number*|| and ||*area*|| both are possible characteristics of Typical. In this conflict, Typical again becomes the object of investigation. This results in a more pronounced use of Typical. While there still is no full definition, there are criteria for its correct form: <*Typical is an area that at most covers half the data*> and <*Typical can be signified by a number, if the area is small*>.

Some minutes later in the discussion, Rebecca tackles the question whether one is allowed to omit data points that could be seen as exceptions when creating report sheets.

61 R Well, you can do that, but it depends. You have to make sure it fits. If you do it like here *[points to own report sheet]* you should not consider the isolated cases […] because then it gets imprecise. But if the typical area was the same on both sides, I think you can do that.

While there still is no full definition of Typical, the use of Typical has been extended. Typical not only functions as a description of the 'relevant' part of the data to be used in comparisons, but also as a tool for determining whether ||*exceptions*|| have to be considered: <*If the Typical area of two distributions is the same, one has to take exceptions into account*>. Typical is not only a tool for describing these exceptions (as it is in Turn 22 for comparison of distributions), but rather in deciding whether to consider values external to Typical in comparison of distributions. It has become a tool for use in a new kind of problem.

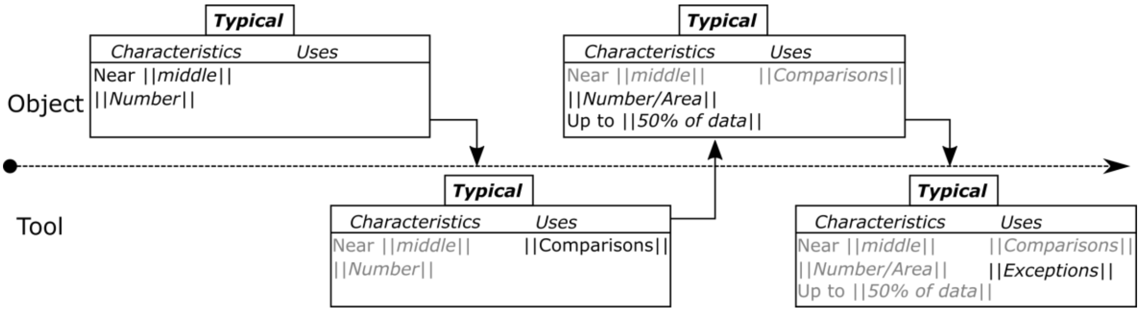

Fig. 5: Consolidating Typical by switching between Object and Tool

Throughout this episode, Typical expands its use and properties and touches on a wide range of aspects of regular statistical measures (Fig. 5): A measure that describes an area (Turn 43) in the middle of the distribution (Turn 2), consisting of no more than half the data points (Turn 43), indicating the location of the most dense area (Turn 3), to be used in comparing distributions (Turn 24) and in identifying data points diverging from a central tendency (Turn 61) indeed shows many similarities to properties and use of box plots. Under this perspective, box plots do not seem as challenging a concept as sometimes indicated (cf. Bakker, Biehler, & Konold, 2004).

CONCLUSION

Consistent to previous cycles, the students in the presented case study adopted the informal measure of Typical to characterize parts of the data as rule and exception. At the end of the interview, Typical stood for an area located in the middle of the distribution characterizing the 'relevant' part of the data, while also describing a tendency towards the part of the distribution where density is highest. This supplied the students with a concept-rich language for capturing ideas relating to statistical concepts such as centre, spread and skewness. After conceptualizing their measure through numbers, as laid out by the report sheets, they were able to use their informal measure of Typical as data-based backing for inferences about differences in the distributions of arctic sea ice.

Contrary to the original description of the tool-object dialectic characterizing the emergence of mathematical concepts on a macro level, no clear progression from use as tool to object was found on the micro level. The learning processes are rather characterized by cyclic phases of use as tool and object, even starting at the investigation of Typical as an object. In each phase, either the definition of Typical or its adequate uses are expanded. Task setting, conflicting filled-in report sheets, and challenges by the researcher each play a part in initiating and mediating these phases. Still it is important to note that the students always keep the agency of their investigation, and Typical stays an informal measure. While this case study focussed only on two students, in the context of the whole design research project similar processes have been observed with all ten students from this cycle. Giving students opportunities to change the roles of their measures thus seems to be a promising way to build informal inferential reasoning.

One of the main reasons for embracing informal statistical inference was to enable students to participate in statistical investigation. Drawing on informal measures takes this approach even one step further, as students' reasoning before being taught regular measures is taken into account. However, using regular measures in drawing informal statistical inferences is still a goal of statistics education. The next steps in research will be to broaden the investigation to additional students and to further investigate the link between individual and regular statistical measure.

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